Gauge theory in 3D and its boundary theories

3D Mirror symmetry
has 2D mirror symmetry on boundary

"Coherent sheaf side" $\overset{?}{\longleftrightarrow}$ "Holomorphic Fukaya Thry"
" Rozansky-Witten theory — Fueter theory ?

Q What are topologically gauged TQFTs ? — pure gauge theory
On bdry of 3D top gauge theory

Meta-theorem:
Pure 3D top gauge Theory $\longleftrightarrow$ RW theory for the H^* Toda integrable system
(LG version) $\longleftrightarrow$ K^* finite diff. Toda system

Old challenge: extend ↗ down to points
X ↘
KRS category for holomorphic symplectic mflds?
D2 of this = BTC DCoh(X)
with structure constructed
from hol str + sympl form.

I have a proposal for Integrable systems
(Based on Rel. field theory)

2. Toda integrable systems (Toda, Kostant, ..)

Bezrukavnikov, Finkelberg, Mirković

G cpt Lie group $\longrightarrow$ $H_*^G(\Omega G) \longleftarrow H_*^G(pt)$

$K_*^G(\Omega G) \longleftarrow K_*^G(pt)$ } Toda bases

Hopf algebras

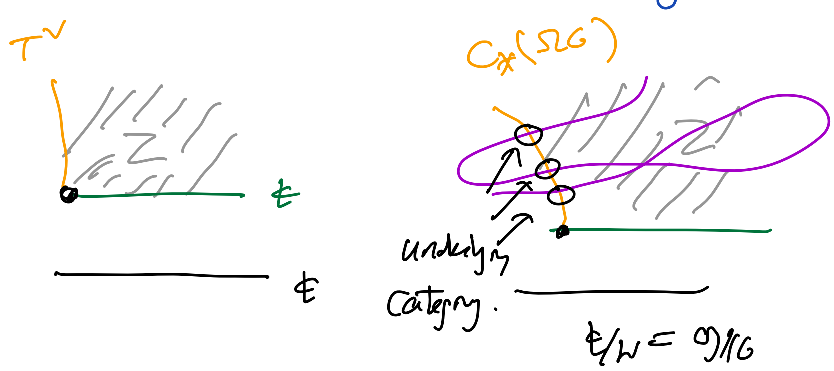
Abelian ✓

Group schemes over the bases, symplectic mflds,

For $G = T$, get $\mathbb{C}^\vee[T^*T^\vee] \longrightarrow \mathbb{C}[T^*, T^\vee]$

geom. $\qquad\qquad \mathrm{Sym}\,\mathfrak{t}^* = H^*(BT)$

[Description in terms of Langlands dual]



sheaves of categories w/ Lagrangian support in Toda space.

Categories w. topological G action

(with really nice properties, eg

HdR sequence collapses

CY condition?

3. Primeval 3D mirror symmetry — EM duality

Finite gp F

Morita equiv.

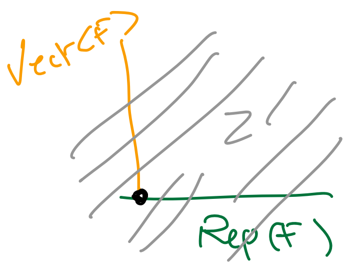
$$(\mathrm{Vect}\langle F\rangle, *) \equiv (\mathrm{Rep}(F), \otimes)$$

Drinfeld center

$$\begin{tikzcd}
 & \mathrm{Vect}\langle F\rangle \arrow[dr] & \\
Z \arrow[ur] \arrow[rr] \arrow[dr] & & \mathrm{Vect} \\
 & \mathrm{Rep}(F) \arrow[ur] &
\end{tikzcd}$$

BTC TC Cat

$$\mathrm{Vect}\langle F\rangle \otimes_Z \mathrm{Rep}(F) = \mathrm{Vect}$$



Point carries full info so gives complete equivalence.

Topological reps of G,

$(\mathrm{Vect}\langle G/\hat{G}\rangle, *)$ $(\mathrm{Rep}(G/\hat{G}), \otimes)$

$$(\mathrm{Vect}\langle B\mathfrak{t}\rangle, *) \underset{\text{localization}}{\overset{\text{after}}{\equiv}} (\mathrm{Rep}(B\mathfrak{t}), \otimes)$$

$(S\mathfrak{t}\text{-modules}, \otimes)$

$\left.\begin{matrix}(\Lambda\mathfrak{t}[1]\text{-mod}, *) \\ (S\mathfrak{t}^*[-2]\text{-mod}, \otimes)\end{matrix}\right\}$ Koszul dual

E_2 HH^* (localized Drinfeld center)

$(\mathrm{Sym}\,\mathfrak{t} \otimes \mathrm{Sym}\,\mathfrak{t}^*[-2], \otimes)$

braided str. from sympl. form.

4. What is $\mathrm{Rep}(G/\hat{G})$?

One answer is the CCC

$G \ltimes \mathrm{Sym}\, \mathfrak{g}^*$, $W = \xi_a(\delta_1) \otimes \xi^a$ $\quad \xi_a$ basis of G

$e^\vee : \mathfrak{g} \longrightarrow G$ exponential

This is a tensor category

$G = T$: $\coprod_\lambda \mathfrak{t}$, $W : \xi \mapsto \langle \lambda | \xi \rangle$

λ weight $\quad$ Tensor structure: pointwise on each sheet addition of labels

Morally: $DZ^{\nearrow} = T^*\mathfrak{t}$ / vertical translation by characters

$= T^* T^\vee$.

Moral statement: Toda space = uncompleted DZ of the tensor cat given by CCC.

If A is an algebra or category with $G/\hat{G}$ action

$$\begin{array}{cccl} 0 & \mathfrak{g} & \xrightarrow{\mathcal{L}} & HZ^1(A) \\ & \| & \circlearrowleft & \uparrow \\ -1 & \mathfrak{g} & \longrightarrow & HH^0(A) \\ & & \searrow & \text{need not be linear} \end{array}$$

DGLA morphism defined from action of $G/\hat{G}$ on algebra or category [L_∞ maps allowed]

Eg $G/\hat{G}$ acts on Vect by the quadratic forms on $\mathfrak{g} \longrightarrow$ 2D Top YM.

Other sanity checks

$G = S^1$, let it act trivially on A

intersecting trivialization $\longrightarrow$ curving of the category

This defines a $B\mathbb{Z}$-action on the category

(Preygel) (auto of Id, e^W)

Thm Tate FP category $=$ $MF(A; W)$.

Very special case of FP construction using CCC. //

Moral calculations fiber of D2 over the base CCC tensor cat
$=$ fiber of Toda system

Pick $\Sigma \in \mathfrak{g}$, centralizer $H \subset G$

$$\mathrm{Vect} \underset{\mathrm{Rep}(H/\hat{H})}{\otimes} \mathrm{Vect} = \mathrm{Vect} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}(H) \underset{\mathrm{Rep}(H/\hat{H})}{\otimes} \mathrm{Vect}$$

$$= \mathrm{Vect} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}(\hat{H}) = \mathrm{Vect}\langle H/\hat{H} \rangle$$

$\parallel$

module cat becomes $\equiv$ by monodromy rep on H

(fcns on dual gp $H^\vee$ if $H = T$)

---

5. What changes when adding matter?

Quaternionic reps of G, E. $E_{\mathfrak{g}} = V \oplus V^\vee$

$G \curvearrowright QH(E)$. ← holomorphic Fukaya 2-cat for E?

When $E = V \oplus V^\vee$, toy model $= \mathrm{End}(\mathcal{F}(V)) \rtimes G$

When E not polarized,

break symmetry to T and restore the Weyl group.

Speculative explanation:

Consider $G \times_T E$, polarize that, form Fukaya category,

Limit $z_i \to \infty$: G/T simplifies to sum of trivial reps.

Turns out can descend back under Weyl gp.

Theorem For each* quat rep E of G there is a space $\mathcal{C}(G;E) \longrightarrow$ Toda base

almost homogeneous for the Toda group scheme

moreover

- if E is polarized, $\mathcal{C}(G;E) \longleftrightarrow$ Toda system
- if not, birational to torsor of order 2.
- There are multiplications

$$\mathcal{C}(G;E) \times \mathcal{C}(G;F) \longrightarrow \mathcal{C}(G;E\oplus F)$$

over the Toda base

- The $\mathcal{C}(G;E)$ are $H_*^G(\Omega G; \mathcal{H}_E)$

$\hookrightarrow$ multiplication for ΩG.

(Polarized case: Braverman, Finkelberg, Nakajima

NP: different methods Braverman et al

or topological break of ?)

- Concrete construction from Toda space + superpotential of GLSM for a polar half of E.
- Polarized case:

subring of functions on $\mathcal{C}(G;0)$

which remain regular by vertical translation

by a rational Lagrangian section

$(\exp(d\,\Psi_V)$.

. NP case: Pass to Weyl cover of base

Choose a positive half of E

Use GLSM superpotential to deform a

"rational" action of the Weyl gp

Descend back.

This gives functions on $\mathcal{C}(G;E)$.

$\ast$: there are some mod 2 obstructions in NP case

to be removed //